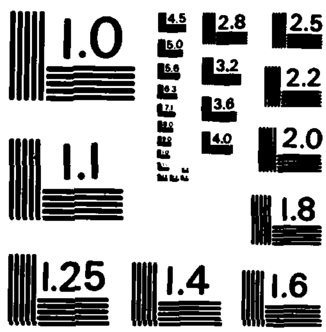
AD-A146 155 SINC METHOD OF SOLUTION OF SINGULAR INTEGRAL EQUATIONS (U) UTAH UNIV SALT LAKE CITY DEPT OF MATHEMATICS F STENGER ET AL. 1984 ARO-19297.8-MA DAAG29-83-K-0012 1/1

UNCLASSIFIED F/G 12/1 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: ARO 19297.8-MA
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): SINC Method of Solution of Singular Integral Equations
5. TYPE OF REPORT & PERIOD COVERED: Technical Report
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Frank Stenger, David Elliott
8. CONTRACT OR GRANT NUMBER(s): DAAG 83 K 0012 29
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Utah, Department of Mathematics, Salt Lake City, Utah 84112
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
11. CONTROLLING OFFICE NAME AND ADDRESS: U. S. Army Research Office, Post Office Box 12211, Research Triangle Park, NC 27709
12. REPORT DATE: 1984
13. NUMBER OF PAGES:
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.

DTIC ELECTE OCT 1 1984 S D B

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES: The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

This document is mainly

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The use of the Whittaker cardinal (or sinc) function for the approximate solution of the complete, one-dimensional, singular integral equation with arbitrary index is discussed. ~~Mostly we shall~~ be concerned with the case when the equation is taken over the arc (-1,1). An indirect method of approximate solution, based on the equivalent Fredholm integral equation, is described. Convergence of the approximate solutions is discussed in some detail and it is shown that the error decays exponentially.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

84 09 25 088

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(2)

ARO 19297.8-MA

SINC METHOD OF SOLUTION OF SINGULAR INTEGRAL EQUATIONS

Frank Stenger*
Department of Mathematics, University of Utah,
Salt Lake City, Utah 84112, USA

David Elliott
Department of Mathematics, University of Tasmania,
Hobart, Tasmania, Australia.

Abstract

The use of the Whittaker cardinal (or sinc) function for the approximate solution of the complete, one-dimensional, singular integral equation with arbitrary index is discussed. Mostly we shall be concerned with the case when the equation is taken over the arc $(-1,1)$. An indirect method of approximate solution, based on the equivalent Fredholm integral equation, is described. Convergence of the approximate solutions is discussed in some detail and it is shown that the error decays exponentially.

1. Introduction

In this paper we propose to investigate the use of the Whittaker cardinal function (or sinc function) for the approximate solution of the singular integral equation

$$a(x)w(x) + \frac{b(x)}{\pi} \int_\Gamma \frac{w(t)dt}{t - x} + \int_\Gamma K(x,t)w(t)dt = f(x), \quad (1.1)$$

for $x \in \Gamma$, where Γ is an open arc which mostly, in this paper, we take to be $(-1,1)$. The functions a, b, K and f of (1.1) are assumed to be given on Γ and it is required to find w, or approximations to w. The first integral appearing in (1.1) is a Cauchy principal value integral defined by

$$\int_\Gamma \frac{w(t)dt}{t - x} = \lim_{\epsilon \to 0+} \int_{\Gamma - \Gamma_\epsilon} \frac{w(t)dt}{t - x}, \quad (1.2)$$

where Γ_ϵ is that part of Γ cut out by a circle of radius ϵ with centre at x, provided that the limit exists.

The theory of equation (1.1) is to be found, for example, in the book by Muskhelishvili [8] and, following him, we shall look for solutions w of (1.1) which are absolutely integrable over $[-1,1]$. Let us define

$$(\mathcal{H} w)(x) = \frac{1}{\pi} \int_\Gamma \frac{w(t)dt}{t - x}, \quad (\mathcal{K} w)(x) = \int_\Gamma K(x,t)w(t)dt, \quad (1.3)$$

then we can rewrite (1.1) as

$$aw + b\mathcal{H} w + \mathcal{K} w = f. \quad (1.4)$$

Suppose that (1.1) has index κ which may be positive, negative or zero. (For the calculation of κ given a and b, see Dow and Elliott [3]). If we define

$$G(x) = (a(x) - ib(x))/(a(x) + ib(x)) \quad (1.5)$$

then the fundamental function Z is defined by

$$Z(x) = \exp\{-(1/2)(\mathcal{H} \log G)(x)\}, \quad (1.6)$$

where we choose appropriate branches of $\log G$ so that it is continuous on $(-1,1)$ and furthermore such that both Z and $1/Z$ are absolutely integrable over $(-1,1)$, see Elliott [4]. As discussed in [3], rather than solve (1.1) for w, it is computationally more convenient to solve for the function F say where

$$w = (Z/r)F, \quad (1.7)$$

the function r being defined by

$$r(x) = (a^2(x) + b^2(x))^{1/2} \quad (1.8)$$

and assumed to be strictly positive for all x in Γ and its end points. We replace (1.4) by the following equation for F,

$$(aZ/r)F + b\mathcal{H}(ZF/r) + \mathcal{K}(ZF/r) = f. \quad (1.9)$$

It is well known (see, for example [8] and [4]) that this equation can be "regularized" so that it is equivalent to

$$F - b\mathcal{H}((1/rZ)\mathcal{K}(ZF/r)) + (a/rZ)\mathcal{K}(ZF/r)$$
$$= (af/rZ) - b\mathcal{H}(f/rZ) + bP_{\kappa-1}, \quad (1.10)$$

where $P_{\kappa-1}$ is an arbitrary polynomial of degree $\kappa-1$, it being understood that $P_{\kappa-1} \equiv 0$ when $\kappa \le 0$. Now (1.10) is a Fredholm integral equation of the second kind and we shall use it as the starting point for all methods of this paper. Thus we shall be describing an "indirect" method for the approximate solution of (1.1).

A review of the use of sinc functions for the approximate solution of various functional equations has been given by Stenger [9], and frequent reference to the results of [9] will be made throughout this paper. However, in §2, we derive a new class of sinc approximations which is particularly suited to the solution of (1.1). For the interval $(-1,1)$, if we let $\mathcal{D}$ denote the eye-shaped region containing $(-1,1)$, see (2.35), then we shall assume that F is analytic in $\mathcal{D}$ but of the class Lip_α in the closure of $\mathcal{D}$; we write $F \in B_\alpha(\mathcal{D})$ (see definition 2.14) and in §2 we derive interpolation and quadrature formulae for such functions.

In §3 we first consider some of the properties of the integral operators arising in equation (1.10) and then we consider its approximate solution. This is based on the Galerkin method but, as we shall show in §3.5, the sinc approximating basis that we use reduces the Galerkin scheme to a Nyström scheme so that it is also a collocation method. The details of the approximation are given in §3.2 and §3.3. In §3.4 we consider the convergence of the approximate solution to the exact solution. In our discretization we obtain a $(2N+3)$ term linear approximation F_N say to F which satisfies

$$\sup_{-1<x<1} |F(x) - F_N(x)| = O(N^{1/2}\exp[-(\pi d\alpha N)^{1/2}]) \quad (1.11)$$

as $N \to \infty$. It is, of course, this exponential (rather than algebraic) decay of the error with N which makes the use of sinc function methods in numerical analysis so attractive. Although we shall not do so here, it may be shown that the rate of decay on the right hand side of (1.11) is optimal (see [2]) in the sense that

* Research supported by U.S. Army Research Contract No. DAAG 83 K 0012.

there is no basis $\{\omega_{kN}\}$, $k = (-N-1)(1)(N+1)$, $N = 1,2,3,\ldots$ such that

$$\sup_{-1<x<1} \left|F(x) - \sum_{k=-N-1}^{N+1} c_k(F)\omega_{kN}(x)\right| = O(p(N,N^{-1})\exp(-\gamma N^{1/2})), \qquad (1.12)$$

as $N\to\infty$, where p is a fixed polynomial in N and N^{-1}, and where $\gamma > (\pi d\alpha)^{1/2}$.

Finally, in §4, we consider a particular example. This paper is intended to provide the foundation for the application of sinc methods to the approximate solution of singular integral equations. Much remains to be done and, in particular, one might mention the study of direct methods for approximate solutions, extensions to other arcs (only briefly mentioned here) and closed contours, and finally the application of these methods to systems of singular integral equations.

2. Integration Formulas for Evaluating Cauchy Principal Value Integrals.

The definitions, notations and results of this section are important to the rest of the paper. We shall derive two families of formulae for a very general contour Γ. These two families reduce to a single family in the case when $\Gamma = R$. The important case of $\Gamma = (-1,1)$ is then given special consideration.

2.1. The Domain $\mathcal{D}_d$ and Approximating Functions on R.

Definition 2.1. Let R denote the real line, $R = (-\infty,\infty)$ let $C = \{z = x + iy: x \in R, y \in R\}$ and let $Z = \{k: k = 0, \pm1, \pm2, \ldots\}$. Let d and h denote positive numbers and let us define

$$\mathcal{D}_d = \{z \in C : |\mathrm{Im} z| < d\}, \qquad (2.1)$$

$$S(k,h)\circ(x) = \frac{\sin[\pi(x-kh)/h]}{[\pi(x-kh)/h]}, \quad k \in Z, \qquad (2.2)$$

$$T(k,h)\circ(x) = \frac{1-\cos[\pi(kh-x)/h]}{[\pi(kh-x)/h]}, \quad k \in Z. \qquad (2.3)$$

2.2 The More General Domain $\mathcal{D}$ and the Conformal Map ϕ.

Definition 2.2. Let $\mathcal{D}$ be a simply connected domain in the complex plane C, and denote by $\partial\mathcal{D}$ the boundary of $\mathcal{D}$. Let a,b ($b \neq a$) be boundary points of $\mathcal{D}$ and let ϕ be a conformal map of $\mathcal{D}$ onto $\mathcal{D}_d$ (see (2.1)) such that $\phi(a) = -\infty$, $\phi(b) = \infty$. Let $\psi = \phi^{-1}$ denote the inverse map and set

$$\Gamma = \{\psi(x) : x \in R\}, \qquad (2.4)$$

$$z_k = \psi(kh), \quad k \in Z. \qquad (2.5)$$

Let $B(\mathcal{D})$ denote the family of all functions F that are analytic in $\mathcal{D}$ and such that

$$N(F,\mathcal{D}) = \int_{\partial\mathcal{D}} |F(z)dz| = \inf_{C\to\partial\mathcal{D}, C\subset\mathcal{D}} \int_C |F(z)dz| < \infty. \qquad (2.6)$$

Theorem 2.3. Let $S(k,h)$ and $T(k,h)$ be defined by (2.2) and (2.3) respectively and let $F \in B(\mathcal{D})$. Then, for all $x \in \Gamma$,

$$\frac{F(x)}{\phi'(x)} - \sum_{k\in Z} \frac{F(z_k)}{\phi'(z_k)} S(k,h)\circ\phi(x) = \frac{\sin[\pi\phi(x)/h]}{2\pi i} \int_{\partial\mathcal{D}} \frac{F(z)dz}{(\phi(z)-\phi(x))\sin[\pi\phi(z)/h]}, \qquad (2.7)$$

$$\int_\Gamma F(x)dx - h\sum_{k\in Z} \frac{F(z_k)}{\phi'(z_k)} = \frac{i}{2}\int_{\partial\mathcal{D}} \frac{F(z)\exp[i(\pi\phi(z)/h)\mathrm{sgn}(\mathrm{Im}\phi(z))]dz}{\sin[\pi\phi(z)/h]}, \qquad (2.8)$$

$$\int_\Gamma F(x)S(k,h)\circ\phi(x)dx - hF(z_k)/\phi'(z_k) = \frac{ih(-1)^k}{2\pi}\int_{\partial\mathcal{D}} \frac{F(z)\exp[i(\pi\phi(z)/h)\mathrm{sgn}(\mathrm{Im}\phi(z))]dz}{\phi(z)-kh}, \qquad (2.9)$$

and

$$\frac{1}{\pi}\int_\Gamma \frac{F(t)dt}{\phi(t)-\phi(x)} - \sum_{k\in Z}\frac{F(z_k)}{\phi'(z_k)}T(k,h)\circ\phi(x) = \frac{1}{2\pi i}\times$$

$$\int_{\partial\mathcal{D}} \frac{F(z)\{\cos[\pi\phi(x)/h] - \exp[i(\pi\phi(z)/h)\mathrm{sgn}(\mathrm{Im}\phi(z))]\}dz}{[\phi(z)-\phi(x)]\sin[\pi\phi(z)/h]}. \qquad (2.10)$$

Moreover, if the left hand sides of equations (2.7), (2.8), (2.9) and (2.10) are denoted by $\eta_1(x)$, η_2, η_3, $\eta_4(x)$ respectively then

$$\left.\begin{aligned}
&|\eta_1(x)| \le N(F,\mathcal{D})/(2\pi d\sinh(\pi d/h)),\ x \in \Gamma,\\
&|\eta_2| < \exp(-\pi d/h)N(F,\mathcal{D})/(2\sinh(\pi d/h)),\\
&|\eta_3| < h\exp(-\pi d/h)N(F,\mathcal{D})/(2\pi d),\\
&|\eta_4(x)| < (1+\exp(-\pi d/h))N(F,\mathcal{D})/(2\pi d\sinh(\pi d/h)),\\
&x \in \Gamma.
\end{aligned}\right\} \qquad (2.11)$$

Proof. See Stenger [9].

Assumption 2.4. In addition to $F \in B(\mathcal{D})$, let us assume that for all $x \in \Gamma$,

$$|F(x)/\phi'(x)| \le C\exp(-\alpha|\phi(x)|), \qquad (2.12)$$

where C,α are positive numbers.

Theorem 2.5. Let $\delta_{i,N}$ denote the left hand sides of equations (2.6+i), $i = 1,2$ and 4, for the case when the infinite sums $\Sigma_{k\in Z}$ are replaced by finite sums $\Sigma_{k=-N}^{N}$. Let F satisfy (2.12). If h is selected by the formula

$$h = (\pi d/\alpha N)^{1/2} \qquad (2.13)$$

then there exist constants C_i which are independent of N such that

$$|\delta_{i,N}| \le C_i N^{1/2}\exp\{-(\pi d\alpha N)^{1/2}\},\ i = 1,2,4. \qquad (2.14)$$

If h is selected by the formula

$$h = (2\pi d/\alpha N)^{1/2} \qquad (2.15)$$

then there exists a constant C_2, such that

$$|\delta_{2,N}| \le C_2\exp\{-(2\pi d\alpha N)^{1/2}\}. \qquad (2.16)$$

If h is selected by the formula

$$h = \gamma/N^{1/2} \qquad (2.17)$$

where γ is a positive constant, then there exist positive constants C' and δ such that

$$|\delta_{i,N}| \le C'\exp(-\delta N^{1/2}),\ i = 1,2,3,4. \qquad (2.18)$$

Proof. See Stenger [9].

We observe that the bounds on $|\eta_1(x)|$ and $|\eta_4(x)|$ in Theorem 2.5 apply uniformly for all $x \in \Gamma$. Moreover the integral $(1/\pi)\int_\Gamma (F(t)/(\phi(t)-\phi(x)))dt$ as well as its approximation via (2.10) approaches zero as x approaches an end-point of Γ. This is not necessarily true for the new approximation which we shall derive next.

Let us define the functions

$$s_k(x) \equiv s(\phi,k,h,x) = \frac{h}{\pi}\cdot\frac{\sin[(\pi/h)\{\phi(x)-kh\}]}{x - z_k}, \qquad (2.19)$$

$$t_k(x) \equiv t(\phi,k,h,x) = \frac{h}{\pi}\cdot\frac{\cos[(\pi/h)\{\phi(x)-kh\}]-1}{x - z_k}, \qquad (2.20)$$

where we intend to use the abbreviated notation when the roles of ϕ and h are understood. From equations (2.2), (2.3), (2.19) and (2.20) it follows that

$$\left.\begin{aligned} S(k,h)\circ\phi(z_\ell) &= \begin{cases} 0, & \text{if } k \neq \ell, \\ 1, & \text{if } k = \ell, \end{cases} \\ T(k,h)\circ\phi(z_\ell) &= \begin{cases} (1-(-1)^{k-\ell})/(\pi(k-\ell)), & \text{if } k\neq\ell, \\ 0, & \text{if } k = \ell; \end{cases} \end{aligned}\right\} \qquad (2.21)$$

and

$$\left.\begin{aligned} s_k(z_\ell) &= \begin{cases} 0, & \text{if } k \neq \ell, \\ \phi'(z_\ell), & \text{if } k = \ell, \end{cases} \\ t_k(z_\ell) &= \begin{cases} h(1-(-1)^{k-\ell})/(\pi(z_k-z_\ell)), & \text{if } k \neq \ell, \\ 0, & \text{if } k = \ell. \end{cases} \end{aligned}\right\} \qquad (2.22)$$

<u>Theorem 2.6.</u> Let $F \in B(\mathcal{D})$. Then for all $x \in \Gamma$, $x \neq a,b$

$$\epsilon_1(x) \equiv F(x) - \sum_{k\in Z}\frac{F(z_k)}{\phi'(z_k)}s_k(x)$$

$$= \frac{\sin(\pi\phi(x)/h)}{2\pi i}\int_{\partial\mathcal{D}}\frac{F(z)dz}{(z-x)\sin[\pi\phi(z)/h]}, \qquad (2.23)$$

$$\epsilon_2 \equiv \int_\Gamma F(x)s_k(x)dx - hF(z_k)$$

$$= \frac{ih(-1)^k}{2\pi}\int_{\partial\mathcal{D}}\frac{F(z)\exp[i(\pi\phi(z)/h)\,\mathrm{sgn}\,\mathrm{Im}\,\phi(z)]}{z - z_k}dz, \qquad (2.24)$$

$$\epsilon_3(x) \equiv \frac{1}{\pi}\int_\Gamma\frac{F(t)dt}{t-x} - \sum_{k\in Z}\frac{F(z_k)}{\phi'(z_k)}t_k(x)$$

$$= \frac{1}{2\pi i}\int_{\partial\mathcal{D}}\frac{F(z)\{\cos[\pi\phi(x)/h]-\exp[i(\pi\phi(z)/h)\,\mathrm{sgn}\,\mathrm{Im}\,\phi(z)]\}dz}{(z-x)\sin[\pi\phi(z)/h]}. \qquad (2.25)$$

Moreover,

$$|\epsilon_1(x)| \le N(F,\mathcal{D},x)/(2\pi\sinh(\pi d/h)), \qquad (2.26)$$

$$|\epsilon_2| \le (h/2\pi)\exp(-\pi d/h)N(F,\mathcal{D},z_k), \qquad (2.27)$$

$$|\epsilon_3(x)| \le N(F,\mathcal{D},x)[1+\exp(-\pi d/h)]/(2\pi\sinh(\pi d/h)), \qquad (2.28)$$

where

$$N(F,\mathcal{D},x) = \int_{\partial\mathcal{D}}|F(z)/(z-x)dz|. \qquad (2.29)$$

<u>Proof.</u> To prove (2.23) let us, in (2.7), replace x by t and $F(x)$ by $[(\phi(t)-\phi(x))/(t-x)]F(t)$ to give

$$\frac{F(t)(\phi(t)-\phi(x))}{\phi'(t)(t-x)} - \sum_{k\in Z}\frac{F(z_k)}{\phi'(z_k)}\cdot\frac{(kh-\phi(x))}{(z_k-x)}S(k,h)\circ\phi(x)$$

$$= \frac{\sin[\pi\phi(x)/h]}{2\pi i}\int_{\partial\mathcal{D}}\frac{F(z)[\phi(z)-\phi(x)]dz}{[\phi(z)-\phi(t)](z-x)\sin[\pi\phi(z)/h]}, \qquad (2.30)$$

this equation holding for any $F \in B(\mathcal{D})$ and all interior points x and t of Γ. Setting $x = t$ in (2.30) gives (2.23).

Equation (2.24) follows directly from (2.8) if, in (2.8), we replace F by $s_k F$ and use definition (2.19) for s_k, noting that $\sin[(\pi/h)(\phi(x)-kh)] = (-1)^k\sin[\pi\phi(x)/h]$.

To prove (2.25) we multiply both sides of (2.30) by $\phi'(t)/(\phi(t)-\phi(x))$, integrate over Γ, and use the identities

$$\int_R\frac{\sin(\pi t/h)}{t-z}dt = \pi\exp\{(i\pi z/h)\,\mathrm{sgn}\,\mathrm{Im}\,z\}, \qquad (2.31)$$

$z \in C$, $z \notin R$ and

$$\int_R\frac{\sin(\pi t/h)}{t-x}dt = \pi\cos(\pi x/h), \qquad (2.32)$$

$x \in R$, from which the result follows.

Finally, the error bounds (2.26), (2.27) and (2.28) follow directly by bounding the contour integrals in (2.23), (2.24) and (2.25) respectively. #

We remark that under our assumptions on F the right hand sides of (2.23) and (2.25) may become unbounded as x approaches an end point of Γ. Hence (2.23) to (2.25) must be interpreted to be accurate in the sense of a relative error. An absolute bound is possible i.e. (2.23) and (2.25) hold for all x on Γ if $M(F,\mathcal{D}) \equiv \sup_{x\in\Gamma} N(F,\mathcal{D},x) < \infty$, see (2.29). In general we must exercise caution in the evaluation of the approximating sums in (2.23) and (2.25). For example, if $x = z_\ell$, then $\epsilon_1(z_\ell) = 0$ which poses no problem. On the other hand if we define

$$(\mathcal{H}F)(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{F(t)dt}{t-x}, \qquad (2.33)$$

then $(\mathcal{H}F)(z_\ell)$ is not necessarily zero and we have, by (2.25)

$$(\mathcal{H}F)(z_\ell) \approx \frac{h}{\pi}\sum_{k\neq\ell}\frac{F(z_k)[1-(-1)^{k-\ell}]}{\phi'(z_k)(z_k-z_\ell)}. \qquad (2.34)$$

The evaluation of this sum to within a relative error of δ can be carried out by means of Algorithm 2.7 (see below). The approximation of $F(x)$, $x \neq z_\ell$ by (2.23) can be carried out by a similar algorithm.

<u>Algorithm 2.7.</u> Evaluation of the sum in (2.34) to within δ relative error.

$k = \ell - 1$
$S_1 = 2F(z_k)/[\phi'(z_k)(z_k-z_\ell)]$
$U = V = 1;\ \ W = |S_1|$
$k = k - 2$
$T = 2F(z_k)/[\phi'(z_k)(z_k-z_\ell)]$
$S_1 = S_1 + T$
$U = V,\ V = W,\ W = |T|$
$(U + V + W)/|S_1| : \delta$
$(\ge)$

(<) $k = \ell + 1$

$S_2 = 2F(z_k)/[\phi'(z_k)(z_k - z_\ell)]$

$U = 1,\ V = 1,\ W = |S_2|$

$k = k + 2$

$T = 2F(z_k)/[\phi'(z_k)(z_k - z_\ell)]$

$S_2 = S_2 + T$

$U = V,\ V = W,\ W = |T|$

$(U + V + W)/|S_2| : \delta$

(≥)

(<) $S = (h/\pi)(S_1 + S_2)$

In order to replace the infinite sums in Theorem 2.6 by finite sums, we consider three special cases of the transformation ϕ.

Ex.2.8. $\Gamma = (-1,1)$. In this case choose

$$\left.\begin{aligned} &\phi(z) = \log\left(\frac{1+z}{1-z}\right), \quad z_k = \frac{e^{kh}-1}{e^{kh}+1}, \ k \in Z, \\ &\mathcal{D} = \left\{z \in C : \left|\arg\left(\frac{1+z}{1-z}\right)\right| < d\right\}, \ 0 < d < \pi. \end{aligned}\right\} \quad (2.35)$$

Ex.2.9. $\Gamma = (0,\infty)$. Choose

$$\left.\begin{aligned} &\phi(z) = \log z, \quad z_k = e^{kh}, \quad k \in Z, \\ &\mathcal{D} = \{z \in C : |\arg z| < d\} \ 0 < d < \pi. \end{aligned}\right\} \quad (2.36)$$

Although the mapping (2.36) suffices for many problems over $(0,\infty)$, it is unsuitable for those for which the corresponding functions to be approximated are unbounded in the region $\mathcal{D}$ of (2.36). The following transformation is suitable in the case when these functions are analytic and bounded in a strip containing the interval $(0,\infty)$.

Ex.2.10. $\Gamma = (0,\infty)$. Here we choose

$$\left.\begin{aligned} &\phi(z) = \log(\sinh z), z_k = \log[e^{kh}+(1+e^{2kh})^{1/2}], k \in Z \\ &\mathcal{D} = \{z : |\arg(\sinh z)| < d\}, \ 0 < d < \pi. \end{aligned}\right\} \quad (2.37)$$

Graphical illustrations of the regions $\mathcal{D}$ of (2.35) to (2.37) are given in [9].

Lemma 2.11. Under any of the transformations (2.35) to (2.37) we have

$$\sup_{x \in \Gamma} |s_k(x)/\phi'(z_k)| \le e^h, \quad (2.38)$$

$$\sup_{x \in \Gamma} |t_k(x)/\phi'(z_k)| < e^h. \quad (2.39)$$

Proof. To prove (2.38) we consider first x to be in the interval $[z_{k-1}, z_{k+1}]$ and then x to be in the remainder of Γ. For $z_{k-1} \le x \le z_{k+1}$, we have

$$|\sin\{(\pi/h)[\phi(x)-kh]\}| < (\pi/h)|\phi(x)-kh|. \quad (2.40)$$

Hence, on this interval, for any $k \in Z$

$$\left|\frac{s_k(x)}{\phi'(z_k)}\right| < \left|\frac{\phi(x)-kh}{\phi'(z_k)(x-kh)}\right|$$

$$< \max_{\xi \in [z_{k-1}, z_{k+1}]} \left|\frac{\phi'(\xi)}{\phi'(z_k)}\right| \le e^h, \quad (2.41)$$

for each of the transformations (2.35) to (2.37).

On the interval $\Gamma \backslash [z_{k-1}, z_{k+1}]$, we use the inequality $|\sin\{(\pi/h)[\phi(x)-kh]\}| \le 1$, then for any $k \in Z$

$$\left|\frac{s_k(x)}{\phi'(z_k)}\right| < \max_{x=z_{k-1}, z_{k+1}} \frac{h/\pi}{|\phi'(z_k)(x-z_k)|}$$

$$= \max_{\zeta_1 \in [(k-1)h,(k+1)h]} \frac{1/\pi}{|\phi'(z_k)\psi'(\zeta_1)|},$$

see Definition (2.2).

$$\therefore \left|\frac{s_k(x)}{\phi'(z_k)}\right| < \max_{\xi \in [z_{k-1}, z_{k+1}]} \frac{1}{\pi}\left|\frac{\phi'(\xi)}{\phi'(z_k)}\right| \le \frac{1}{\pi}e^h, \quad (2.42)$$

from (2.41), for each of the transformations (2.35) to (2.37).

Consider now (2.39). On the interval $[z_{k-1}, z_{k+1}]$ we have

$$\left|\frac{t_k(x)}{\phi'(z_k)}\right| = \left|\frac{h[1-\cos\{(\pi/h)[\phi(x)-kh]\}]}{\pi\phi'(z_k)(x-z_k)}\right|$$

$$= \left|\frac{\int_{z_k}^{x} \phi'(t)\sin\{(\pi/h)[\phi(t)-kh]\}dt}{\phi'(z_k)(x-z_k)}\right|$$

$$\le \max_{\xi \in [z_{k-1}, z_k]} \left|\frac{\phi'(\xi)}{\phi'(z_k)}\right|, \text{ for } k \in Z.$$

Hence as in (2.41) we have

$$|t_k(x)/\phi'(z_k)| \le e^h. \quad (2.43)$$

Finally on $\Gamma \backslash [z_{k-1}, z_{k+1}]$ we have

$|1 - \cos\{(\pi/h)[\phi(x)-kh]\}| \le 2$ so that, for $k \in Z$,

$$\left|\frac{t_k(x)}{\phi'(z_k)}\right| \le \max_{x=z_{k-1}, z_{k+1}} \frac{(2h/\pi)}{|\phi'(z_k)(x-z_k)|} \le \frac{2}{\pi}e^h, \quad (2.44)$$

where the last inequality follows by proceeding as in the derivation of (2.42). ∎

Assumption 2.12. Let us assume, in addition to $F \in B(\mathcal{D})$, that for some positive constants α and C

$$|F(z)| \le C e^{-\alpha|\phi(z)|}, \ z \in \mathcal{D}. \quad (2.45)$$

We note that (2.45) is equivalent to

$$|F(z)| \le C|1-z^2|^\alpha, \ z \in \mathcal{D}, \quad (2.46)$$

for the case of (2.35); to

$$|F(z)| \le C|z/(1+z^2)|^\alpha, \ z \in \mathcal{D}, \quad (2.47)$$

for the case (2.36); and to

$$|F(z)| \le C|z e^{-z}|^\alpha, \ z \in \mathcal{D}, \quad (2.48)$$

for the case of (2.37).

Theorem 2.13. If Assumption 2.12 is satisfied and if ϕ is any one of the transformations defined in (2.35) to (2.37), then by taking $h = [\pi d/\alpha N]^{1/2}$, there exist constants C_1 and C_2, independent of N, such that for all $x \in \Gamma$

$$\left|F(x) - \sum_{k=-N}^{N} (F(z_k)/\phi'(z_k))s_k(x)\right| \le C_1 N^{1/2}\exp(-(\pi d\alpha N)^{1/2}), \quad (2.49)$$

and

$$\left|\frac{1}{\pi}\int_\Gamma \frac{F(t)dt}{t-x} - \sum_{k=-N}^{N} \frac{F(z_k)}{\phi'(z_k)} t_k(x)\right| \le C_2 N^{1/2} \exp(-(\pi d\alpha N)^{1/2}). \tag{2.50}$$

<u>Proof.</u> If Assumption (2.12) is satisfied then it follows for each of the transformations ϕ of (2.35) to (2.37) that $N(F,\mathcal{D}) \equiv \sup_{x\in\Gamma} N(F,\mathcal{D},x) < \infty$, see (2.29). Then for $\varepsilon_j(x)$, $j = 1,3$ defined by (2.26) and (2.28) it follows that

$$|\varepsilon_j(x)| \le A \exp(-\pi d/h), \tag{2.51}$$

where A is independent of h. Hence, for (2.49), we have on using (2.38) that

$$\left|F(x) - \sum_{k=-N}^{N} F(z_k)(s_k(x)/\phi'(z_k))\right|$$

$$\le A e^{-\pi d/h} + e^h \sum_{|k|>N} |F(z_k)|$$

$$\le A e^{-\pi d/h} + 2C e^h \sum_{k=N+1}^{\infty} e^{-\alpha kh}, \text{ from (2.46)}$$

$$= A e^{-\pi d/h} + 2C e^h e^{-\alpha(N+1)h}/(1-e^{-\alpha h})$$

$$< A e^{-\pi d/h} + 2C e^h e^{-\alpha Nh}/(\alpha h).$$

Inequality (2.51) now follows on choosing $h = (\pi d/(\alpha N))^{1/2}$. The proof for $|\varepsilon_3(x)|$ follows similarly, using (2.39). ∎

<u>2.3 The Special Case of $\Gamma = (-1,1)$.</u> In this case the transformation ϕ, the region $\mathcal{D}$ and the points z_k, $k \in Z$ are defined as in (2.35). The region $\mathcal{D}$ is bounded by two circular arcs intersecting at the end points ± 1 at an angle d say (see Figure 4.2 of [9]).

If $F \in C[-1,1]$ we define LF by

$$(LF)(z) = (1-z)F(-1)/2 + (1+z)F(1)/2. \tag{2.52}$$

Then, see (2.33),

$$(\mathcal{H}\, LF)(x) = (1/\pi)[F(-1)+F(1)] - (1/\pi)\phi(x)(LF)(x). \tag{2.53}$$

<u>Definition 2.14.</u> Let $\mathcal{D}$ be defined as in (2.35), let $0 < \alpha \le 1$, and let $B_\alpha(\mathcal{D})$ denote the family of all functions that are analytic in $\mathcal{D}$ and of class Lip_α in $\bar{\mathcal{D}}$, the closure of $\mathcal{D}$.

<u>Lemma 2.15.</u> Let $F \in B_\alpha(\mathcal{D})$, then the function G where

$$G = F - LF \tag{2.54}$$

is analytic in $\mathcal{D}$ and satisfies (2.46) for all $z \in \bar{\mathcal{D}}$.

<u>Proof.</u> We have

$$\left.\begin{aligned} |F(z) - F(1)| &< K|1-z|^\alpha, \ z \in \bar{\mathcal{D}}, \\ |F(z) - F(-1)| &< K|1+z|^\alpha, \ z \in \bar{\mathcal{D}}, \end{aligned}\right\} \tag{2.55}$$

where K is the Lipschitz constant of F. Now if $z \in \mathcal{D}$ and $|1-z| < 1$, then $|1+z| \ge 1$ so that for all such z, $K|1-z|^\alpha \le K|1-z^2|^\alpha$. Similarly if $|1+z| < 1$, then $|1-z| \ge 1$ and the lemma follows. ∎

<u>Theorem 2.16.</u> Let ϕ, z_k and $\mathcal{D}$ be defined as in (2.35). Let $F \in B_\alpha(\mathcal{D})$, let h be chosen so that

$$h = (\pi d/(\alpha N))^{1/2}, \tag{2.56}$$

and let $\mathcal{H}(LF)$ be defined by (2.53). Then there exists a constant C, independent of N, such that for all $x \in [-1,1]$,

$$\left|F(x)-(LF)(x) - \sum_{k=-N}^{N} [F(z_k)-(LF)(z_k)]S(k,h)\circ\phi(x)\right| \le C N^{1/2} \exp(-(\pi d\alpha N)^{1/2}), \tag{2.57}$$

$$\left|F(x)-(LF)(x) - \sum_{k=-N}^{N} [F(z_k)-(LF)(z_k)](s_k(x)/\phi'(z_k))\right| \le C N^{1/2} \exp(-(\pi d\alpha N)^{1/2}), \tag{2.58}$$

$$\left|\frac{1}{\pi}\int_{-1}^{1} \frac{F(t)dt}{t-x} - \mathcal{H}(LF)(x) - \sum_{k=-N}^{N} [F(z_k)-(LF)(z_k)]T(k,h)\circ\phi(x)\right| \le C N^{1/2} \exp(-(\pi d\alpha N)^{1/2}), \tag{2.59}$$

and

$$\left|\frac{1}{\pi}\int_{-1}^{1} \frac{F(t)dt}{t-x} - \mathcal{H}(LF)(x) - \sum_{k=-N}^{N} [F(z_k)-(LF)(z_k)](t_k(x)/\phi'(z_k))\right| \le C N^{1/2} \exp(-(\pi d\alpha N)^{1/2}). \tag{2.60}$$

<u>Proof.</u> This is a consequence of Lemma 2.15 and Theorems 2.5 and 2.13. ∎

<u>Remark 2.17.</u> If h is selected by

$$h = \gamma/N^{1/2} \tag{2.61}$$

where γ is a positive constant, then there exist positive constants C_1 and δ, independent of N, such that the right hand sides of (2.57) to (2.60) may be replaced by $C_1 \exp(-\delta N^{1/2})$.

3. <u>The Integral Equation</u>

<u>3.1 The Operator T.</u>

Following the discussion of §1 (see (1.10)), let T be an operator defined by

$$TF = -\left\{ \frac{a}{rZ}\mathcal{H}\left(\frac{ZF}{r}\right) - b\mathcal{H}\left(\frac{1}{rZ}\mathcal{H}\left(\frac{ZF}{r}\right)\right)\right\}. \tag{3.1}$$

Let us define $k(x,t)$ by

$$k(x,t) = K(x,t)Z(t)/r(t). \tag{3.2}$$

<u>Assumption 3.1.</u> Let $\mathcal{D}$ and $B_\alpha(\mathcal{D})$ be defined as in Definition 2.14 and let us assume that:

(a) for some fixed $x_1 \in (-1,1)$,

$$\beta \equiv \int_{-1}^{1} |k(x_1,t)|dt < \infty ; \tag{3.3}$$

(b) $k(\cdot,t)$ is analytic in $\mathcal{D}$ for each fixed $t \in (-1,1)$;

(c) $k(\cdot,t)$ satisfies a Lip_α condition, i.e.

$$|k(x,t) - k(y,t)| \le C(t)|x-y|^\alpha \tag{3.4}$$

for all $t \in (-1,1)$, for all $x,y \in \bar{\mathcal{D}}$ and where $C(t)$ is such that

$$\gamma \equiv \int_{-1}^{1} C(t)dt < \infty . \tag{3.5}$$

<u>Lemma 3.2.</u> Let Q be defined by

$$(QF)(x) = \int_{-1}^{1} k(x,t)F(t)dt. \tag{3.6}$$

Then Q is a compact operator mapping $L^\infty[-1,1]$ into $B_\alpha(\mathcal{D})$.

Proof. Let $F \in L^\infty[-1,1]$ be given, and let us set

$$\|F\|_\infty = \sup_{-1<x<1} |F(x)|. \tag{3.7}$$

Then the inequality

$$|k(x,t)| \le |k(x_1,t)| + |k(x,t) - k(x_1,t)| \tag{3.8}$$

combined with (3.3), (3.4) and (3.5) yields

$$|(QF)(x)| \le (\beta + \eta^\alpha \gamma)\|F\|_\infty \tag{3.9}$$

where

$$\eta = \max_{\substack{x\in[-1,1]\\ y\in\partial\mathcal{D}}} |x-y| = \begin{cases} 2, & \text{if } 0 < d < \pi/2, \\ 2/\sin d, & \text{if } \pi/2 < d < \pi, \end{cases} \tag{3.10}$$

from which we have that

$$\|Q\| < \beta + \eta^\alpha \gamma. \tag{3.11}$$

Moreover, we clearly have that QF is analytic in $\mathcal{D}$ and furthermore, for all $x,y \in \bar{\mathcal{D}}$ we have, by (3.4) and (3.5) that

$$\begin{aligned} |(QF)(x)-(QF)(y)| &\le \int_{-1}^{1} |k(x,t)-k(y,t)|\,|F(t)|dt \\ &\le |x-y|^\alpha \cdot \|F\|_\infty \int_{-1}^{1} C(t)dt \\ &= \gamma|x-y|^\alpha \|F\|_\infty. \end{aligned} \tag{3.12}$$

That is, QF maps a bounded set $\{F \in L^\infty[-1,1] : \|F\|_\infty < A\}$ into a family of functions that are analytic and uniformly bounded in $\bar{\mathcal{D}}$ and of class Lip_α in $\bar{\mathcal{D}}$. This proves Lemma 3.2. #

Lemma 3.3. Let a and b be in $B_\alpha(\mathcal{D})$ and suppose that

$$|r| = |(a^2+b^2)^{1/2}| > \delta > 0 \text{ in } \bar{\mathcal{D}}. \tag{3.13}$$

Then the operator T, see (3.1), is a compact operator mapping $L^\infty[-1,1]$ into $B_\alpha(\mathcal{D})$.

Proof. Setting $f = QF$, it suffices in view of Lemma 3.2 to show that the operator R defined by

$$Rf = \frac{af}{rZ} - b\mathcal{H}\left(\frac{f}{rZ}\right) \tag{3.14}$$

is a compact operator mapping $B_\alpha(\mathcal{D})$ into $B_\alpha(\mathcal{D})$. Now, see [4], the solution to the equation

$$aZg/r + b\mathcal{H}(Zg/r) = f \tag{3.15}$$

having index κ is given by

$$g = Rf + bP_{\kappa-1} \tag{3.16}$$

where $P_{\kappa-1}$ is an arbitrary polynomial of degree $\kappa-1$ ($P_{\kappa-1} \equiv 0$ if $\kappa \le 0$). Also under the given conditions $g \in \mathrm{Lip}_\alpha[-1,1]$ whenever $f \in \mathrm{Lip}_\alpha[-1,1]$, see [5].

In order to show that $Rf \in B_\alpha(\mathcal{D})$, let β be an arbitrary constant in the range $-d < \beta < d$, and let $A(\beta)$ denote the circular arc

$$A(\beta) = \{\zeta \in \mathcal{D} : \arg(1+\zeta)/(1-\zeta) = \beta\}. \tag{3.17}$$

Let us fix $z \in \mathcal{D}$ and β such that $\beta < \arg(1+z)/(1-z)$ and consider the function

$$w(z) = \frac{b(z)}{\pi}\int_{A(\beta)} \frac{h(t)dt}{t-z}\ ;\ h = \frac{f}{rZ}. \tag{3.18}$$

On letting $z \to \zeta \in A(\beta)$ we get

$$w(\zeta) = b(\zeta)\left\{ih(\zeta) + \frac{1}{\pi}\int_{A(\beta)} \frac{h(t)dt}{t-\zeta}\right\}. \tag{3.19}$$

Taking $\beta = -d$ in (3.18) we see that $w(z)$ is analytic in $\mathcal{D}$. Next, with ζ an arbitrary point of $\mathcal{D}$, we see from (3.19) that since both $w(\zeta)$ and $b(\zeta)h(\zeta)$ are analytic in $\mathcal{D}$ so is

$$v(\zeta) \equiv \frac{b(\zeta)}{\pi}\int_{A(\beta)} \frac{h(t)dt}{t-\zeta}. \tag{3.20}$$

We next show that $Rf \in \mathrm{Lip}_\alpha(\bar{\mathcal{D}})$. To this end, it suffices to show that the solution g to

$$\frac{a(x)Z(x)g(x)}{r(x)} + \frac{b(x)}{\pi}\int_{A(\beta)} \frac{Z(t)g(t)dt}{r(t)(t-x)} = f(x),\ x \in A(\beta) \tag{3.21}$$

is of class $\mathrm{Lip}_\alpha[A(\beta)]$ whenever $f \in \mathrm{Lip}_\alpha[A(\beta)]$. Let us set

$$\sigma = \frac{1}{\sin\beta} - \frac{1}{\tan\beta},\ x = \frac{\xi - i\sigma}{1-i\sigma\xi},\ t = \frac{\tau-i\sigma}{1-i\sigma\tau}. \tag{3.22}$$

Under this transformation (3.21) becomes

$$\frac{a_1(\xi)Z_1(\xi)g_1(\xi)}{r_1(\xi)} + \frac{b_1(\xi)}{\pi}\int_{-1}^{1} \frac{Z_1(\tau)g_1(\tau)d\tau}{r_1(\tau)(\tau-\xi)} = f_1(\xi), \tag{3.23}$$

$-1 < \xi < 1$, where $a_1(\xi), b_1(\xi), Z_1(\xi), r_1(\xi)$ are related to $a(x), b(x), Z(x), r(x)$ respectively by a relation of the form

$$\left.\begin{aligned} &u_1(\xi) = u\left(\frac{\xi-i\sigma}{1-i\sigma\xi}\right),\ \text{whereas} \\ &f_1(\xi) = f\left(\frac{\xi-i\sigma}{1-i\sigma\xi}\right)\cdot\frac{1}{1-i\sigma\xi},\ g_1(\xi) = g\left(\frac{\xi-i\sigma}{1-i\sigma\xi}\right)\cdot\frac{1}{1-i\sigma\xi} \end{aligned}\right\} \tag{3.24}$$

Note that a_1 and b_1 are analytic continuations into $\mathcal{D}$ of a and b respectively so that they are in the class $\mathrm{Lip}_\alpha[-1,1]$ and satisfy $|a_1^2 + b_1^2|^{1/2} \ge \delta > 0$ on $[-1,1]$. Also, since $f \in \mathrm{Lip}_\alpha(\bar{\mathcal{D}})$ it follows from (3.24) that $f_1 \in \mathrm{Lip}_\alpha[-1,1]$. Consequently, in view of (3.15) and (3.16), the solution g_1 to (3.23) is of class $\mathrm{Lip}_\alpha[-1,1]$.

This proves that Rf is analytic in $\mathcal{D}$ and of class $\mathrm{Lip}_\alpha(\bar{\mathcal{D}})$. Finally, let us set

$$\|R\|_{\mathcal{D}} = \sup_{\substack{g\in B_\alpha(\mathcal{D}) \\ \max_{(z\in\bar{\mathcal{D}})}|g(z)| \le 1}} |(Rg)(z)| \tag{3.25}$$

Clearly we must have $\|R\|_{\mathcal{D}} < \infty$, for if $\|R\|_{\mathcal{D}} = \infty$, then as a consequence of the fact that the supremum on the right hand side of (3.25) is taken over a compact family, there would exist a $g \in B_\alpha(\mathcal{D})$ with $|g(z)| \le 1$ in $\bar{\mathcal{D}}$ such that $Rg \notin B_\alpha(\mathcal{D})$, contradicting $Rg \in B_\alpha(\mathcal{D})$ for all $g \in B_\alpha(\mathcal{D})$. #

Remark 3.4. We have also shown in the above proof that if $F \in L^\infty[-1,1]$ then since $TF = -RQF$, we have

$$\sup_{z\in\mathcal{D}} |(TF)(z)| = \sup_{z\in\mathcal{D}} |(RQF)(z)| \le \|R\|_{\mathcal{D}} \|Q\| \|F\|_\infty, \tag{3.26}$$

so that

$$\|T\|_{\mathcal{D}} \le \|R\|_{\mathcal{D}} \|Q\| < \infty, \tag{3.27}$$

where $\|F\|_\infty$ is defined in (3.7) and $\|Q\|$ is bounded as in (3.11).

Also, if K is the smallest constant such that

$$|(TF)(x) - (TF)(y)| \le K|x-y|^{\alpha} \tag{3.28}$$

for all F with $\|F\|_{\infty} \le 1$, it follows that we must have $0 < K < \infty$ and that for arbitrary $F \in L^{\infty}[-1,1]$,

$$|(TF)(x) - (TF)(y)| \le K\|F\|_{\infty}|x-y|^{\alpha}. \tag{3.29}$$

3.2 Approximation on [-1,1].

With $\mathcal{D}$ and ϕ defined as in (2.35) let us also write, for a given positive integer N,

$$\left.\begin{aligned}
&\zeta_k = (e^{kh}-1)/(e^{kh}+1), \quad k = -N(1)N, \\
&\zeta_{-N-1} = -1, \quad \zeta_{N+1} = 1\ ; \\
&\psi_k(z) = s_k(z)/\phi'(z_k), \quad k = -N(1)N, \\
&\psi_{-N-1}(z) = (1-z)/2 - \sum_{k=-N}^{N} (1-\zeta_k)\psi_k(z)/2, \\
&\psi_{N+1}(z) = (1+z)/2 - \sum_{k=-N}^{N} (1+\zeta_k)\psi_k(z)/2.
\end{aligned}\right\} \tag{3.30}$$

Given $F \in L^{\infty}[-1,1]$ we also set

$$\mathcal{P}_N F = \sum_{k=-N-1}^{N+1} F(\zeta_k)\psi_k. \tag{3.31}$$

Remark 3.5. We have, with LF defined by (2.52)

$$F - \mathcal{P}_N F = F - LF - \sum_{k=-N}^{N} G(\zeta_k)\psi_k \tag{3.32}$$

so that if $F \in B_{\alpha}(\mathcal{D})$ then

$$\|F - \mathcal{P}_N F\|_{\infty} = \begin{cases} O(\exp(-\delta N^{1/2})), & \text{if } h = \gamma/N^{1/2}, \\ O(N^{1/2}\exp(-(\pi d\alpha N)^{1/2})), & \end{cases} \tag{3.33}$$

if $h = (\pi d/\alpha N)^{1/2}$, where γ and $\delta = \delta(\gamma)$ are positive constants.

3.3 Approximation of the Integral Equation.

We shall obtain an approximate solution of the integral equation

$$\mathcal{F} - T\mathcal{F} = (af)/(rZ) - b\mathcal{H}(f/(rZ)) + bP_{r-1} \equiv g \tag{3.34}$$

where P_{r-1} is a polynomial of degree $r-1$. Let us replace (3.34) by the equation

$$\mathcal{F}_N - T_N\mathcal{F}_N = g_N \tag{3.35}$$

where

$$\left.\begin{aligned}
&\mathcal{F}_N = \sum_{k=-N-1}^{N+1} c_k\psi_k, \\
&g_N = \mathcal{P}_N g \quad \text{and} \quad T_N = \mathcal{P}_N T.
\end{aligned}\right\} \tag{3.36}$$

Lemma 3.6. *Let T, T_N be considered as operators on $L^{\infty}[-1,1]$. Then*

$$\|T-T_N\| = \sup_{\substack{-1<x<1 \\ \|F\|_{\infty}\le 1}} |(TF)(x) - (T_N F)(x)|$$

$$= \begin{cases} O(\exp(-\delta N^{1/2})), & \text{if } h = \gamma/N^{1/2}, \\ O(N^{1/2}\exp(-(\pi d\alpha N)^{1/2})), & \end{cases} \tag{3.37}$$

if $h = [\pi d/(\alpha N)]^{1/2}$, where γ and δ are positive numbers such that given $\gamma > 0$, there exists $\delta = \delta(\gamma) > 0$.

Proof. Let us write

$$v = TF, \quad v_N = \mathcal{P}_N TF = T_N F. \tag{3.38}$$

Then by Lemma 3.3, $v \in B_{\alpha}(\mathcal{D})$ and, by Remark 3.4, $\|T\|_{\mathcal{D}} < \infty$ so that

$$\sup_{z \in \mathcal{D}} |v(z)| \le \|T\|_{\mathcal{D}} \|F\|_{\infty} < \infty. \tag{3.39}$$

Hence, setting

$$u = v - Lv, \tag{3.40}$$

where L is defined in (2.52), we have

$$\sup_{z\in\mathcal{D}} |u(z)| \le \sup_{z\in\mathcal{D}} |v(z)| + \sup_{z\in\mathcal{D}} |(Lv)(z)| \le 3\|T\|_{\mathcal{D}}\|F\|_{\infty}. \tag{3.41}$$

Hence, by Lemma 2.15,

$$|u(z)| \le C|1-z^2|^{\alpha}, \quad z \in \mathcal{D}, \tag{3.42}$$

where C is a positive constant. Indeed, in view of (3.29), we may take $C = K\|F\|_{\infty}$. Combining with (2.23) we have for all $x \in (-1,1)$

$$\left|u(x) - \sum_{k\in\mathbb{Z}} u(z_k)s_k(x)/\phi'(z_k)\right| \le K\|F\|_{\infty} J/(2\pi\sinh(\pi d/h)) \tag{3.43}$$

where

$$J = \sup_{-1<x<1} \int_{\partial\mathcal{D}} \frac{|1-z^2|^{\alpha}|dz|}{|z-x|} < \infty. \tag{3.44}$$

Furthermore, by Lemma 2.11 and (3.42) with $C = K\|F\|_{\infty}$, we have

$$\begin{aligned}
\left|\sum_{|k|>N} \frac{u(z_k)}{\phi'(z_k)} s_k(x)\right| &\le 2e^{h}K\|F\|_{\infty} \sum_{k=N+1}^{\infty} (1-z_k^2)^{\alpha} \\
&= 2^{1+2\alpha}K\|F\|_{\infty}\, e^{h} \sum_{k=N+1}^{\infty} \frac{e^{\alpha kh}}{(1+e^{kh})^{2\alpha}} \\
&\le 2^{1+2\alpha}K\|F\|_{\infty}\, e^{h} \sum_{k=N+1}^{\infty} e^{-\alpha kh} \\
&\le \frac{2^{1+2\alpha}e^{h}}{\alpha h} e^{-\alpha Nh} K\|F\|_{\infty}.
\end{aligned} \tag{3.45}$$

Combining (3.45) and (3.43) we get, for all $x \in (-1,1)$

$$\left|u(x) - \sum_{k=-N}^{N} u(z_k)\psi_k(x)\right| \le \left\{\frac{J}{2\pi\sinh(\pi d/h)} + \frac{2^{1+2\alpha}e^{-(\alpha N-1)h}}{\alpha h}\right\} K\|F\|_{\infty}. \tag{3.46}$$

That is, from (3.32), (3.40) and (3.38),

$$\|v - \mathcal{P}_N v\|_{\infty} = \|TF - \mathcal{P}_N TF\|_{\infty} \le \left\{\frac{J}{2\pi\sinh(\pi d/h)} + \frac{2^{1+2\alpha}e^{-(\alpha N-1)h}}{\alpha h}\right\} K\|F\|_{\infty}. \tag{3.47}$$

Equation (3.37) now follows from (3.47) according to the selection of h. #

3.4. Convergence of approximations when $(I-T)^{-1}$ exists.

The following result is due to Banach (see, for example, [7]).

Lemma 3.7. Let X be a Banach space, with T and $(I-T)^{-1}$ continuous linear operators on X. Let T_N be a compact

linear operator on X such that

$$\|(I-T)^{-1}\| \cdot \|T-T_N\| < 1. \tag{3.48}$$

Then $(I-T_N)^{-1}$ exists and

$$\|(I-T_N)^{-1}\| \le \frac{\|(I-T)^{-1}\|}{1-\|(I-T)^{-1}\|\,\|T-T_N\|}. \tag{3.49}$$

__Lemma 3.8.__ Let X, T and T_N be defined as in Lemma 3.7. Then

$$(I-T)^{-1} - (I-T_N)^{-1}$$

$$= (I-T)^{-1}(T-T_N)[I+(I-T)^{-1}(T-T_N)]^{-1}(I-T)^{-1}, \tag{3.50}$$

so that for a given $g \in X$, if (3.48) is satisfied then

$$\|(I-T)^{-1}g - (I-T_N)^{-1}g\|$$

$$\le \frac{\|T-T_N\|\cdot\|(I-T)^{-1}\|^2\cdot\|g\|}{1-\|T-T_n\|\cdot\|(I-T)^{-1}\|}. \tag{3.51}$$

__Proof.__ See Ikebe [6].

We can now prove the following theorem.

__Theorem 3.9.__ If $g \in B_\alpha(\mathcal{D})$ and if $(I-T)^{-1}$ exists as a bounded operator on $L^\infty[-1,1]$ then

$$\|(I-T)^{-1}g - (I-T_N)^{-1}\mathcal{P}_N g\|_\infty$$

$$= \begin{cases} O(N^{\frac{1}{2}}\exp(-(\pi d\alpha N)^{\frac{1}{2}}), & h = (\pi d/\alpha N)^{\frac{1}{2}}, \\ O(\exp(-\delta N^{\frac{1}{2}})), & h = \gamma/N^{\frac{1}{2}}, \end{cases} \tag{3.52}$$

where γ and δ are positive numbers.

__Proof.__ Since $g \in B_\alpha(\mathcal{D})$ we have $g \in L^\infty[-1,1]$; furthermore $\mathcal{P}_N g \in L^\infty[-1,1]$. Since

$$(I-T)^{-1}g - (I-T_N)^{-1}\mathcal{P}_N g$$

$$= \{(I-T)^{-1}g - (I-T_N)^{-1}g\} + (I-T_N)^{-1}(g-\mathcal{P}_N g),$$

(3.52) follows from (3.51), (3.49) and (3.33). #

__3.5. Galerkin-Nyström Quadrature Approximation.__

In addition to (a), (b) and (c) of Assumption 3.1 let us also assume that

(d) for each fixed $x \in (-1,1)$, $k(x,\cdot)$ is in $B(\mathcal{D})$ (see definition 2.2), and moreover for $-1 < t < 1$ and $x \in [-1,1]$

$$|k(x,t)| \le C_2|1-t^2|^{\alpha-1} \tag{3.53}$$

where C_2 is a positive constant and where α is the same as in Assumption 3.1.

__Lemma 3.10.__ If $F \in B(\mathcal{D})$ and if $|F(t)| \le C_2(1-t^2)^{\alpha-1}$ on $(-1,1)$ then

$$\left|\int_{-1}^{1} F(t)dt - h\sum_{j=-N-1}^{N+1} \frac{2e^{jh}}{(1+e^{jh})^2} F\left(\frac{e^{jh}-1}{e^{jh}+1}\right)\right|$$

$$= \begin{cases} O(\exp(-(\pi d\alpha N)^{\frac{1}{2}})), & \text{if } h = (\pi d/(\alpha N))^{\frac{1}{2}}, \\ O(\exp(-\delta N^{\frac{1}{2}})), & \text{if } h = \gamma/N^{\frac{1}{2}}, \end{cases} \tag{3.54}$$

where γ is a positive constant and $\delta = \delta(\gamma) > 0$.

__Proof.__ See Stenger [9, Example 4.8]. #

Similarly we can use (2.25) to approximate $(\mathcal{H}F)(z_\ell)$ on $(-1,1)$ this approximation taking the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{F(t)dt}{t-z_\ell} \doteq \frac{h(1+e^{\ell h})}{\pi}\sum_{j\ne\ell}\frac{e^{jh}(1-(-1)^{j-\ell})}{(1+e^{jh})(e^{jh}-e^{\ell h})}F\left(\frac{e^{jh}-1}{e^{jh}+1}\right), \tag{3.55}$$

where the sum may be evaluated to within a relative error of the same order of magnitude as that on the right hand side of (3.54) by using Algorithm 2.7.

We shall next show that under our assumptions on $k(x,t)$, our Galerkin approximation scheme (3.30) and (3.35) for solving (3.34) may be readily reduced to the Nyström scheme (see, for example, [1] and [6]). To illustrate, we observe that while $\mathcal{P}_N F$ interpolates F at z_k, $k = -N(1)N$, as well as at ± 1, the absolute value of the error introduced on replacing $(\mathcal{P}_N F)(z_{N+1})$ by $(\mathcal{P}_N F)(1)$ is

$$|(\mathcal{P}_N F)(z_{N+1}) - (\mathcal{P}_N F)(1)| = |c_{N+1}-c_{-N-1}|(1-z_{N+1})/2$$

$$< |c_{N+1}-c_{-N-1}|\exp[-(N+1)h], \tag{3.56}$$

this being of smaller order than the error of interpolation given by (3.33), which we have chosen to ignore. Similarly $|(\mathcal{P}_N F)(z_{-N-1}) - (\mathcal{P}_N F)(-1)|$ is also bounded by the right hand side of (3.56).

Similarly, the quadrature formula in (3.54) yields

$$\mathcal{K}\left(\frac{Z}{r}\mathcal{P}_N F\right)(x) \doteq h\sum_{j=-N-1}^{N+1} 2e^{jh}k(x,z_j)c_j/(1+e^{jh})^2 - \epsilon$$

where

$$\epsilon = 2he^{-3(N+1)h}[k(x,z_{-N-1}) - k(x,z_{N+1})] \times$$

$$\times [c_{-N-1}-c_{N+1}]/(1+e^{-(N+1)h})^3, \tag{3.57}$$

so that ϵ is of much smaller order than the order of quadrature error in (3.54). Hence we can safely set $\epsilon = 0$ to get a much simpler approximation, which is the Nystrom approximation.

Similarly if $f \in B_\alpha(\mathcal{D})$ then both $u = af/(rZ)$ and $v = b\mathcal{H}\{f/(rZ)\}$ are in $B_\alpha(\mathcal{D})$, and the evaluation of $\mathcal{H}\{f/(rZ)\}(x)$ via Algorithm 2.7 is straightforward for $x = z_{\pm(N+1)}$ but may be impossible for $x = \pm 1$. Hence it is also convenient here to replace $x = \pm 1$ by $x = z_{\pm(N+1)}$, respectively. The modulus of the error in making these replacements is given by

$$|v(1)-v(z_{N+1})| \le K|1-z_{N+1}|^\alpha \le 2^\alpha K\exp(-\alpha(N+1)h),$$

so that it is of smaller order than the interpolation error in (3.33).

The exact linear system corresponding to (3.35) is

$$(\underline{I} - \underline{T})\underline{c} = \underline{g} \tag{3.58}$$

where

$$\underline{c} = (c_{-N-1},\ldots,c_{N+1})^T, \quad \underline{g} = (g_{-N-1},\ldots,g_{N+1})^T.$$

$\underline{I}$ here denotes the unit matrix of order $(2N+3)$ and $\underline{T}$ is a square matrix $(t_{m,n})$, $m,n = (-N-1)(1)(N+1)$ of order $(2N+3)$. We have

$$g_m = \frac{a(\zeta_m)f(\zeta_m)}{r(\zeta_m)Z(\zeta_m)} - b(\zeta_m)\left\{\mathcal{H}\left(\frac{f}{rZ}\right)(\zeta_m) + P_{K-1}(\zeta_m)\right\} \tag{3.59}$$

for $m = (-N-1)(1)(N+1)$, and

$$t_{m,n} = -\int_{-1}^{1}\left\{\frac{a(\zeta_m)k(\zeta_m,t)}{r(\zeta_m)Z(\zeta_m)} - b(\zeta_m)\mathcal{H}[k(\cdot,t)](\zeta_m)\right\}\psi_n(t)\,dt \tag{3.60}$$

for $m,n = (-N-1)(1)(N+1)$, and ζ_m is defined by (3.30). In view of the above discussion, we replace the system (3.58) - (3.60) by a perturbed linear system of the same order, given by

$$(\underset{\sim}{I} - \hat{\underset{\sim}{T}})\,\hat{\underset{\sim}{c}} = \hat{\underset{\sim}{g}} \tag{3.61}$$

where $\underset{\sim}{I}$ is as before, $\hat{\underset{\sim}{c}} = (\hat{c}_{-N-1},\ldots,\hat{c}_{N+1})^T$ and the entries $\hat{g}_m, m = (-N-1)(1)(N+1)$ of the vector $\hat{\underset{\sim}{g}}$ and $\hat{t}_{m,n}$ of the matrix $\hat{\underset{\sim}{T}}$, where $m,n = (-N-1)(1)(N+1)$, are given by

$$\hat{g}_m = \frac{a(z_m)f(z_m)}{r(z_m)Z(z_m)} - b(z_m)\left\{\mathcal{H}\left[\frac{f}{rZ}\right](z_m) + P_{\kappa-1}(z_m)\right\}, \tag{3.62}$$

$$\hat{t}_{m,n} = -\frac{2h\,e^{nh}}{(1+e^{nh})^2}\left\{\frac{a(z_m)k(z_m,z_n)}{r(z_m)Z(z_m)} - b(z_m)\mathcal{H}[k(\cdot,z_n)](z_m)\right\}. \tag{3.63}$$

The singular integrals $\mathcal{H}[\;\;]$ in (3.62) and (3.63) are approximated via the substitution of the formula on the right hand side of (3.55) into Algorithm 2.7.

Bounds on the difference $\|\underset{\sim}{c}-\hat{\underset{\sim}{c}}\|$ of the solutions of (3.58) and (3.61) can be obtained by the well known results of linear algebra see, for example, Wilkinson [10], where we note that in this case when $h = (\pi d/(\kappa N))^{\frac{1}{2}}$ the perturbations on the coefficients of the matrix and the right hand side are $O(N^{\frac{1}{2}}\exp(-(\pi d\kappa N)^{\frac{1}{2}}))$.

4. An Example

The algorithm described by (3.61) - (3.63) has been applied to the equation

$$-(1-x^2)^{\frac{1}{2}}w(x) + \frac{x}{\pi}\int_{-1}^{1}\frac{w(t)\,dt}{t-x} + \frac{1}{\pi}\int_{-1}^{1}\frac{w(t)\,dt}{t+x+3}$$

$$= -1 + (t^2 + 6t + 8)^{-\frac{1}{2}}. \tag{4.1}$$

For this equation we have $r(x) = 1$, the fundamental function $Z(x) = 2(1-x^2)^{-\frac{1}{2}}$ and the index $\kappa = 2$. If we choose the particular solution so that $F = 0.5$ at any two points of $(-1,1)$ then we find that $F(x) = 0.5$ for all $x \in (-1,1)$. By choosing $h = \pi/(2N)^{\frac{1}{2}}$ and various values of N we have obtained 1 significant figure in the approximate solution when $N = 4$, two significant figures when $N = 8$, three when $N = 16$ and five when $N = 32$. In fact we find that

$$\max_{k=-N(1)N} |F(z_k) - F_N(z_k)| = 3.0243\exp\{-2.2556N^{\frac{1}{2}}\}, \tag{4.2}$$

approximately.

5. References

[1] Atkinson K.E., Extensions of the Nyström method for the numerical solution of linear integral equations, U.S. Army Math. Res. Center, Tech. Rept. No.686, Madison, 1966.

[2] Bourchard H.C. and Höllig W.C., N-Width and entropy of H_p-Classes in $L_q[-1,1]$, to appear.

[3] Dow M.L. and Elliott D., The numerical solution of singular integral equations over (-1,1), SIAM J.Numer. Anal., 16, 1979, pp.115-134.

[4] Elliott D., The numerical treatment of singular integral equations - a review, from "Treatment of Integral Equations by Numerical Methods", ed. by C.T.H. Baker and G.F.Miller, Academic Press, London, 1982, pp.297-312.

[5] Elliott D., Rates of convergence for the method of classical collocation for solving singular integral equations, SIAM J. Numer. Anal., 21 1984, to appear.

[6] Ikebe Y., The Galerkin method for the numerical solution of Fredholm integral equations of the second kind, SIAM Review, 14, 1972, pp.465-491.

[7] Kantorovich L.V. and Akilov G.P., Functional Analysis, Pergamon Press, Oxford, 1982.

[8] Muskhelishvili N.I., Singular Integral Equations, Nordhoff, Groningen, 1953.

[9] Stenger F., Numerical methods based on Whittaker cardinal, or sinc functions, SIAM Review, 23, 1981, pp.165-224.

[10] Wilkinson J.H., Rounding Errors in Algebraic Processes, H.M.S.O., London, 1963.

EN